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SUBSEQUENCE GENERATORS FOR ERGODIC GROUP TRANSLATIONS†

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ABSTRACT

Let T be an ergodic translation on a compact abelian group. For every infinite set of integers $\{n_i\}$ and $\varepsilon > 0$ there is a set A of measure less than ε such that $\{T^{n_i}A\}$ generates the σ-algebra of measurable sets.

The primary aim of this paper is to prove that for any ergodic translation T on a compact abelian group, infinite set of integers $\{n_i\}$ and $\varepsilon > 0$ there is a set A of measure less than ε such that $\{T^{n_i}A\}$ generates the σ-algebra of measurable sets.

The following fact, which is proved in [1], will be used.

THEOREM 1. *Let T be a $1-1$ aperiodic bimeasurable measure-preserving transformation on a probability space. For every infinite set of integers W and $\varepsilon > 0$ there is a set A of measure less than ε such that $\bigcup_{n \in W} T^n A$ is the entire space.*

Throughout this paper T will denote an ergodic translation on a compact abelian group G (that is, $T(x) = xg$ for some fixed $g \in G$) and $\{n_i\}$ will denote a sequence of integers (indexed by the positive integers) in which no integer is repeated. We assume that G is a Lebesque space. Multiplicative notation will be used for G. Given two measurable subsets of G, E and F, let $EF = \{xy : x \in E$ and $y \in F\}$, E^c denote the complement of E, $E - F$ denote $E \cap F^c$, $E \Delta F$ denote $(E - F) \cup (F - E)$, $\mu(E)$ denote the measure of E and ∂E denote the boundary of E. Let

$$\mathcal{S} = \{D \subseteq G : \mu(\partial D) = 0\}.$$

The proof of the following lemma is straightforward and omitted.

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LEMMA 2. *The collection $\mathcal{S}$ is closed under finite union and intersection, complementation, and translation by any group element.*

LEMMA 3. *There is a countable collection of sets in $\mathcal{S}$ which separates points.*

PROOF. A compact abelian group with an ergodic translation has for its dual group a subgroup of the circle group with the discrete topology [6, p. 40]. Let Π denote the circle group and let Γ denote the dual group of G. Since G is a Lebesque space it is separable, so Γ is countable (if Γ were uncountable its dual would not be separable; yet, by the Pontryagin Duality Theorem [6, p. 27], G is Γ's dual).

If $x \neq y$, the Pontryagin Duality Theorem implies there is a $\gamma \in \Gamma$ for which $\gamma(x) \neq \gamma(y)$. If $\gamma(G) \neq \Pi$, then $\gamma(G)$ is a closed subgroup, hence finite. If a, $b \notin \gamma(G)$, $\gamma(x) \in [a, b]$ and $\gamma(y) \notin [a, b]$, then $\partial(\gamma^{-1}[a, b]) = \emptyset$ and $\gamma^{-1}[a, b]$ separates x and y.

If $\gamma(G) = \Pi$, then $\mu(\gamma^{-1}(a)) = 0$ for all $a \in \Pi$. (Suppose $\mu(\gamma^{-1}(a)) > 0$ for some $a \in \Pi$. Since $\gamma(G) = \Pi$, for each $b \in \Pi$ there is a $g \in G$ with $\gamma(g) = a^{-1}b$; hence $\mu(\gamma^{-1}(b)) = \mu(\gamma^{-1}(a)g) = \mu(\gamma^{-1}(a))$. However, since $\{\gamma^{-1}(b): b \in \Pi\}$ are pairwise disjoint, they cannot all have positive measure, so all must have measure zero.) If $\gamma(x) \in [a, b]$ and $\gamma(y) \notin [a, b]$, then, since $\partial(\gamma^{-1}[a, b]) = \gamma^{-1}(a) \cup \gamma^{-1}(b)$, $\gamma^{-1}[a, b]$ is in $\mathcal{S}$ and separates x and y.

Let K be a countable dense subset of Π. Then by the preceding,

$$\{\gamma^{-1}[a, b]: a, b \in K, \gamma \in \Gamma, \text{ and } a, b \notin \gamma(G) \text{ if } \gamma(G) \neq \Pi\}$$

is a countable collection of sets in $\mathcal{S}$ which separates points. Q.E.D.

For a Lebesque space, any countable collection of sets which separates points generates the σ-algebra of measurable sets [5, p. 22]; hence Lemmas 2 and 3 imply the following.

COROLLARY 4. *For every measurable set B and $\varepsilon > 0$ there is a $D \in \mathcal{S}$ with $\mu(B \,\Delta\, D) < \varepsilon$.*

LEMMA 5. *For every $D \in \mathcal{S}$ and $\varepsilon > 0$ there is a neighborhood $\mathcal{O}$ of the identity such that $\mu(\mathcal{O}D \cap \mathcal{O}D^c) < \varepsilon$.*

PROOF. Let $\mathcal{O}_n$, $n \in \mathbf{N}$ be a nested sequence of open sets which form a basis at the identity. Then

$$\mu(\partial D) = \mu\left(\bigcap_{n \in \mathbf{N}} (\mathcal{O}_n D \cap \mathcal{O}_n D^c)\right) = \lim_{n \to \infty} \mu(\mathcal{O}_n D \cap \mathcal{O}_n D^c). \qquad \text{Q.E.D.}$$

Suppose T is a translation by g. A sequence $\{n_i\}$ will be called convergent for

T if g^{n_i} converges to some $g_0 \in G$ as i goes to infinity. Since the groups being considered are compact, the following lemma is true.

LEMMA 6. *Every* $\{n_i\}$ *has convergent subsequences for* T.

The following lemma is our reason for considering sets whose boundary has measure zero.

LEMMA 7. *If* $\{n_i\}$ *is convergent for* T, *then for every* $D \in \mathcal{S}$,

$$\lim_{k\to\infty} \mu\left(\bigcup_{i=k}^{\infty} T^{n_i}D - \bigcap_{i=k}^{\infty} T^{n_i}D\right) = 0.$$

PROOF. Let $\varepsilon > 0$. By Lemma 5 there is a neighborhood $\mathcal{O}$ of the origin satisfying $\mu(\mathcal{O}D \cap \mathcal{O}D^c) < \varepsilon$. Suppose $g^{n_i} \to g_0$. Choose $k \in \mathbf{N}$ so that for $i \geqq k$, $g^{n_i} \in \mathcal{O}_{g_0}$. Then $\bigcup_{i=k}^{\infty} T^{n_i}D \subseteq \mathcal{O}Dg_0$ and $\bigcap_{i=k}^{\infty} T^{n_i}D \supseteq G - \mathcal{O}D^c g_0$, whence

$$\bigcup_{i=k}^{\infty} T^{n_i}D - \bigcap_{i=k}^{\infty} T^{n_i}D \subseteq \mathcal{O}Dg_0 - (G - \mathcal{O}D^c g_0) = (\mathcal{O}D \cap \mathcal{O}D^c)g_0$$

so

$$\mu\left(\bigcup_{i=k} T^{n_i}D - \bigcap_{i=k} T^{n_i}D\right) \leqq \mu(\mathcal{O}D \cap \mathcal{O}D^c) < \varepsilon. \qquad \text{Q.E.D.}$$

If $D \notin \mathcal{S}$, the conclusion of Lemma 7 may not hold. In this case all that may be asserted is that $\lim_{i,j\to\infty} \mu(T^{n_i}D - T^{n_j}D) = 0$.

LEMMA 8. *For every* $\{n_i\}$ *and* $\varepsilon > 0$ *there is a natural number* r *and a* $D \in \mathcal{S}$ *with* $\mu(D) < \varepsilon$ *and* $\mu(\bigcup_{i=1}^{r} T^{n_i}D) = 1$.

PROOF. Theorem 1 implies there is a set B and an $r \in \mathbf{N}$ with $\mu(B) < \varepsilon/4$ and $\mu(\bigcup_{i=1}^{r} T^{n_i}B) > 1 - \varepsilon/4$. By Corollary 4 choose $E \in \mathcal{S}$ with $\mu(E \,\Delta\, B) < \varepsilon/4r$. Then $\mu(\bigcup_{i=1}^{r} T^{n_i}E) > 1 - \varepsilon/2$. Let $D = E \cup T^{-n_1}(G - \bigcup_{i=1}^{r} T^{n_i}E)$; Lemma 2 implies $D \in \mathcal{S}$. Q.E.D.

LEMMA 9. *If* $\{n_i\}$ *is convergent for* T, $D \in \mathcal{S}$ *and* $\varepsilon > 0$, *then there is a* $B \in \mathcal{S}$ *satisfying* $\mu(B) < \varepsilon$ *and* $\mu(D \,\Delta\, \bigcup_{i=1}^{\infty} T^{n_i}B) < \varepsilon$.

PROOF. By Lemma 5 there is a neighborhood $\mathcal{O}$ of the identity satisfying

$$\mu(\mathcal{O}D \cap \mathcal{O}D^c) < \varepsilon/4. \tag{1}$$

Suppose $g^{n_i} \to g_0$. Choose $k \in \mathbf{N}$ so that for $i \geqq k$, $g^{n_i} \in \mathcal{O}g_0$. By Lemma 8 choose $E \in \mathcal{S}$ with $\mu(E) < \varepsilon/4k$ and $\mu(\bigcup_{i=1}^{\infty} T^{n_i}E) = 1$. Let $B(Dg_0^{-1}) \cap E$. Then

$$\bigcup_{i=k}^{\infty} T^{n_i}(B) \subseteq \bigcup_{i=k}^{\infty} T^{n_i}(Dg_0^{-1}) \subseteq \mathcal{O}D \tag{2}$$

and

$$(3)\qquad \bigcup_{i=k}^{\infty} T^{n_i}(E-B) \subseteq \bigcup_{i=k}^{\infty} T^{n_i}(D^c g_0^{-1}) \subseteq \mathcal{O}D^c.$$

Since $\mu(\bigcup_{i=k}^{\infty} T^{n_i}E) > 1-\varepsilon/4$, (1), (2), and (3) imply that

$$\mu\left(\mathcal{O}D - \bigcup_{i=k}^{\infty} T^{n_i}B\right) < \varepsilon/2.$$

Thus

$$\mu\left(D \,\Delta \bigcup_{i=1}^{\infty} T^{n_i}B\right) \leqq \mu\left(\bigcup_{i=1}^{k-1} T^{n_i}B\right) + \mu(D\,\Delta\,\mathcal{O}D) + \mu\left(\mathcal{O}D \,\Delta \bigcup_{i=k}^{\infty} T^{n_i}B\right)$$
$$< \varepsilon/4 + \varepsilon/4 + \varepsilon/2. \qquad \text{Q.E.D.}$$

THEOREM 10. *For every $\{n_i\}$ and $\varepsilon > 0$ there is a set A whose measure is less than ε such that $\{T^{n_i}A\}$ generates the σ-algebra of measurable sets.*

PROOF. Choose a subsequence $\{m_i\}$ of the $\{n_i\}$ so that $\{m_i\}$ is convergent for T. We shall construct a set A with $\mu(A) < \varepsilon$ such that $\{T^{m_i}A\}$ generates.

By Corollary 4 a sequence $\{E_n\}$ of sets in $\mathcal{S}$ can be found which generates. Let $\{I_n\}$ be a listing of $\{E_n\}$ in which each E_n occurs infinitely often. Let $\varepsilon > 0$, and for $n \in \mathbf{N}$ let $\varepsilon_n = 2^{-n}\varepsilon$.

Sequences $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ of sets in $\mathcal{S}$ and sequences of positive integers $\{p_n\}$, $\{j_n\}$ and $\{k_n\}$, $n \in \mathbf{N}^+$, will now be defined by induction. Let $A_1 = \emptyset$.

Suppose A_n has been defined, $A_n \in \mathcal{S}$. By Lemma 7 choose a positive integer p_n satisfying

$$(1)\qquad \mu\left(\bigcup_{i=p_n}^{\infty} T^{m_i}A_n - \bigcap_{i=p_n}^{\infty} T^{m_i}A_n\right) < \varepsilon_n$$

and also satisfying $p_n > k_{n-1}$ if $n > 1$.

By Lemma 9 choose $B_n \in \mathcal{S}$ satisfying

$$(2)\qquad \mu(B_n) < \varepsilon_{n+1}p_n^{-1}$$

and

$$(3)\qquad \mu\left(I_n \,\Delta \bigcup_{i=1}^{\infty} T^{m_i}B_n\right) < \varepsilon_{n+1}.$$

Thus by Lemma 7 choose an integer j_n, $j_n > p_n$, which satisfies

$$(4)\qquad \mu\left(I_n \,\Delta \bigcup_{i=p_n}^{j_n} T^{m_i}B_n\right) < \varepsilon_n$$

and

$$\mu\left(\bigcup_{i=j_n}^{\infty} T^{m_i}B_n - \bigcap_{i=j_n}^{\infty} T^{m_i}B_n\right) < \varepsilon_n. \tag{5}$$

By Lemma 9 choose $C_n \in \mathcal{S}$ and an integer k_n, $k_n > j_n$, so that

$$\mu(C_n) < \varepsilon_{n+1} j_n^{-1}, \tag{6}$$

$$\mu\left(I_n \,\Delta \bigcup_{i=1}^{\infty} T^{m_i}C_n\right) < \varepsilon_{n+1} \tag{7}$$

and

$$\mu\left(I_n \,\Delta \bigcup_{i=j_n}^{k_n} T^{m_i}C_n\right) < \varepsilon_n \tag{8}$$

are satisfied. Finally, let $A_{n+1} = (A_n \cup B_n) - C_n$; by Lemma 2, $A_{n+1} \in \mathcal{S}$. Thus, the induction is completed.

Note that for all $n \in \mathbf{N}^+$

$$p_n < j_n < k_n < p_{n+1}. \tag{9}$$

Our set A will be $\bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} A_n$. Note that (2) implies

$$\mu(A) \leqq \sum_{n=1}^{\infty} \mu(B_n) < \sum_{n=1}^{\infty} \varepsilon_{n+1} = \varepsilon/2.$$

For $n \in \mathbf{N}^+$ let $D_n = \bigcup_{i=p_n}^{j_n} T^{m_i}A - \bigcap_{i=j_n}^{k_n} T^{m_i}A$.

We will now proceed to show that for $n \in \mathbf{N}^+$

$$\mu(I_n \,\Delta\, D_n) < 8\varepsilon_n. \tag{10}$$

Note that

$$\bigcup_{i=p_n}^{j_n} T^{m_i}A \supseteq \bigcup_{i=p_n}^{j_n} T^{m_i}B_n - \bigcup_{r=n}^{\infty} \bigcup_{i=p_n}^{j_n} T^{m_i}C_r \tag{11}$$

and by (6) and (9)

$$\mu\left(\bigcup_{r=n}^{\infty} \bigcup_{i=p_n}^{j_n} T^{m_i}C_r\right) < j_n \sum_{r=n}^{\infty} \varepsilon_{r+1} j_r^{-1} < \sum_{r=n}^{\infty} \varepsilon_{r+1} = \varepsilon_n. \tag{12}$$

Inequalities (4), (11), and (12) imply

$$\mu\left(I_n - \bigcup_{i=p_n}^{j_n} T^{m_i}A\right) < \mu\left(I_n - \bigcup_{i=p_n}^{j_n} T^{m_i}B_n\right) + \varepsilon_n < 2\varepsilon_n. \tag{13}$$

Since $C_n - \bigcup_{r=n+1}^{\infty} B_r \subseteq A^c$, inequalities (2), (8), and (9) imply

$$(14)\qquad \mu\left(I_n \cap \left(\bigcap_{i=j_n}^{k_n} T^{m_i}A\right)\right) \leqq \mu(I_n) - \mu\left(I_n \cap \left(\bigcup_{i=j_n}^{k_n} T^{m_i}\left(C_n - \bigcup_{r=n+1}^{\infty} B_r\right)\right)\right)$$

$$\leqq \mu(I_n) + \mu\left(\bigcup_{i=j_n}^{k_n} \bigcup_{r=n+1}^{\infty} T^{m_i}B_r\right)$$

$$- \mu\left(I_n \cap \left(\bigcup_{i=j_n}^{k_n} T^{m_i}C_n\right)\right)$$

$$< \mu(I_n) + k_n \sum_{r=n+1}^{\infty} \varepsilon_{r+1} p_r^{-1} - (\mu(I_n) - \varepsilon_n)$$

$$< 2\varepsilon_n.$$

Inequalities (13) and (14) imply

$$(15)\qquad \mu(I_n - D_n) \leqq \mu\left(I_n - \bigcup_{i=p_n}^{j_n} T^{m_i}A\right) + \mu\left(I_n \cap \left(\bigcap_{i=j_n}^{k_n} T^{m_i}A\right)\right) < 4\varepsilon_n.$$

Since $A \subseteq A_n \cup B_n \cup (\bigcup_{r=n+1}^{\infty} B_r)$ and $A_n - \bigcup_{r=n}^{\infty} C_r \subseteq A$, inequalities (1), (2), (4), (6), (8) and (9) imply

$$(16)\qquad \mu(D_n - I_n) \leqq \mu\left(\left(\bigcup_{i=p_n}^{j_n} T^{m_i}A_n - \bigcap_{i=j_n}^{k_n} T^{m_i}A\right) - I_n\right) + \mu\left(\left(\bigcup_{i=p_n}^{j_n} T^{m_i}B_n\right) - I_n\right)$$

$$+ \mu\left(\bigcup_{i=p_n}^{j_n} \bigcup_{r=n+1}^{\infty} T^{m_i}B_r\right)$$

$$< \mu\left(\bigcup_{i=p_n}^{j_n} T^{m_i}A_n - \bigcap_{i=j_n}^{k_n} T^{m_i}A_n\right)$$

$$+ \mu\left(\left(\bigcap_{i=j_n}^{k_n} T^{m_i}A_n - \bigcap_{i=j_n}^{k_n} T^{m_i}A\right) - I_n\right)$$

$$+ \varepsilon_n + j_n \sum_{r=n+1}^{\infty} \varepsilon_{r+1} p_r^{-1}$$

$$< \varepsilon_n + \mu\left(\left(\bigcup_{i=j_n}^{k_n} \bigcup_{r=n}^{\infty} T^{m_i}C_r\right) - I_n\right) + \varepsilon_n + \varepsilon_{n+1}$$

$$\leqq \mu\left(\left(\bigcup_{i=j_n}^{k_n} T^{m_i}C_n\right) - I_n\right) + \mu\left(\bigcup_{i=j_n}^{k_n} \bigcup_{r=n+1}^{\infty} T^{m_i}C_r\right) + 2\varepsilon_n + \varepsilon_{n+1}$$

$$< \varepsilon_n + k_n \sum_{r=n+1}^{\infty} \varepsilon_{r+1} j_r^{-1} + 2\varepsilon_n + \varepsilon_{n+1} < 4\varepsilon_n.$$

Inequalities (15) and (16) imply that (10) holds.

Since (10) holds for each $n \in \mathbf{N}^+$ and since each set E_n in the generating sequence appears in $\{I_n\}$ infinitely often, each E_n is in the σ-algebra generated by $\{T^{m_i}A\}$, whence $\{T^{m_i}A\}$ generates the σ-algebra of measurable sets. Q.E.D.

A measure-preserving transformation (on a Lebesque space) has sequence entropy zero for every sequence of integers if and only if it has discrete spectrum [4, theorem 4]. An ergodic measure-preserving transformation with discrete spectrum is conjugate to a translation on a compact abelian group [2, p. 48]. Hence as a corollary to Theorem 10 we have the following.

COROLLARY 11. *If an ergodic measure-perserving transformation T has sequence entropy zero for every sequence, then for every $\{n_i\}$ there is a set A with $\mu(A) < \varepsilon$ such that $\{T^{n_i}A\}$ generates the σ-algebra of measurable sets.*

Corollary 11 is a start at extending to arbitrary sequences of integers Kreiger's result [3] concerning entropy and generators.

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